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Similarity based approximate reasoning: fuzzy control

Swapan Raha*, Abul Hossain, Sujata Ghosh

Department of Mathematics, Visva-Bharati, Santiniketan 731235, India

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Abstract

This paper presents an approach to similarity based approximate reasoning that elucidates the connection between similarity and existing approaches to inference in approximate reasoning methodology. A set of axioms is proposed to get a reasonable measure of similarity between two fuzzy sets. The similarity between the fact(s) and the antecedent of a rule is used to modify the relation between the antecedent and the consequent of the rule. An inference is drawn using the well-known projection operation on the domain of the consequent. Zadeh's compositional rule of inference and existing similarity based reasoning techniques are considered for a new similarity based approximate reasoning technique. The proposed mechanism is used to develop a modified fuzzy control system. A new defuzzification scheme is proposed. Simulation results are presented for the well-known inverted pendulum problem.

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1. Introduction

The cognitive process of human reasoning deals with imprecise premises. Traditional two-valued logic and/or multi-valued logics are not effective in handling such reasoning processes. Zadeh developed approximate reasoning methodology to tackle the complex problem [40,42]. The desire to build up a quantitative framework that will allow us to derive an approximate conclusion from imprecise knowledge is the main motivation of the theory of approximate reasoning. Fuzzy logic is the basis of approximate reasoning. A proposition in fuzzy logic is represented by a fuzzy set [38].

A collection of imprecise information given by human experts often forms the basis of a fuzzy system which is represented by fuzzy sets or fuzzy relations. The task of a fuzzy system is to exploit the knowledge acquired by experts over time and model the world with it. A fuzzy system reasons with its knowledge. Different patterns of reasoning in human beings indicate a need for similarity matching, in situations where there is no directly applicable knowledge, to come up with a plausible conclusion. In such cases, the confidence in a conclusion may be determined, based on a degree of similarity between the fact(s) and the antecedent of a rule. We know that fuzzy set theory is based on the fuzzification of the predicate 'belongs to'. The indistinguishability modelled by fuzzy set is computable and this concept cannot be overcome in approximate reasoning [11]. In order to capture this, an inference model should have the required flexibility. Specifically, we need means to handle graded information on one hand and the concept of

* Corresponding author.

E-mail address: raha_swapan@rediffmail.com (S. Raha).

similarity on the other hand. Conventional approximate reasoning does not consider the concept of similarity measure in deriving a consequence. Existing similarity based reasoning methods modify the consequence part of a rule, based on a measure of similarity and therefore, the consequence becomes independent of the conditionals. To satisfy both the requirements simultaneously, we need to integrate conventional approximate reasoning and similarity based reasoning for an adequate theory of similarity based approximate reasoning.

Zadeh introduced the concept of Compositional Rule of Inference (CRI). Let us consider Zadeh's form of inference in approximate reasoning based on the Compositional Rule of Inference: From ' X is A ' and ' (X, Y) is R ' infer ' Y is B '. Symbolically, $B = A \circ R$ which is explicitly given by

$$\mu_B(v) = \sup_{u \in U} \{T(\mu_A(u), \mu_R(u, v))\}; \quad (1)$$

where A is a fuzzy subset of U , B is a fuzzy subset of V and R is a fuzzy binary relation on $U \times V$, U and V being the universes of discourse of the linguistic variables X and Y , T is a T-norm function. CRI scheme is such that for a large class of A , each different from the other, the concluded B remains the same (since we are using sup/inf type of operation). Such relations may also produce significant conclusions from an almost dissimilar pair $\{A, A^*\}$ (it is easy to see that whatever A^* be we can always derive a conclusion using CRI). In [31], the authors proposed an alternative model for similarity based analogical approximate reasoning. Recently in [30], a similar scheme for similarity based reasoning has been propounded. In similarity based reasoning, from a given fact the conclusion is derived based on a measure of similarity between the fact and the antecedent of a rule. As for example, let $U = \{a, b, c, d\}$ and $V = \{u, v, w, x\}$ be the universes of discourse,

$$A = \{(1.0, a), (0.75, b), (0.5, c), (0.25, d)\} \quad \text{and}$$

	u	v	w	x
$R = a$	1.00	0.75	0.50	0.25
b	0.75	1.00	0.75	0.50
c	0.50	0.75	1.00	0.75
d	0.25	0.50	0.75	1.00

then taking $T = \min$ and using CRI we find $B = \{(1.00, u), (0.75, v), (0.75, w), (0.50, x)\}$.

This shows that the linguistic variables X and Y are approximately equal. A careful scrutiny of the relation also says so. It is easy to see that the conclusion B will remain the same if we choose $A' = \{(1.0, a), (0.75, b)\}$ or $A' = \{(1.0, a), (0.75, c), (0.50, d)\}$, which are highly dissimilar to A . Next, if we take $A = \{(1.0, a)\}$ then from R we have $B = \{(1.00, u), (0.75, v), (0.50, w), (0.25, x)\}$ again, if we take $A = \{(1.0, d)\}$ then $B = \{(0.25, u), (0.50, v), (0.75, w), (1.00, x)\}$. This shows that even if the input values are strongly complementary to each other, significant conclusions can be drawn using Zadeh's CRI.

In a rule based system, in general, we deduce a conclusion B' from an observed fact A' and a general rule $A \rightarrow B$ using Sup-T composition as

$$\mu_{B'}(v) = \sup_{u \in U} \{T(\mu_{A'}(u), \mu_R(u, v))\} \quad (2)$$

where $\mu_R(u, v) = T_1(\mu_A(u), \mu_B(v))$. If we choose, for convenience, $T = T_1 = \min$ then from $A = \{(1.0, a), (0.75, b), (0.5, c), (0.25, d)\}$, $B = \{(1.0, a), (0.75, b), (0.5, c), (0.25, d)\}$ and $A' = \{(1.0, a), (0.5625, b), (0.25, c), (0.0625, d)\}$ we find that

	u	v	w	x
$R = a$	1.00	0.75	0.50	0.25
b	0.75	0.75	0.50	0.25
c	0.50	0.50	0.50	0.25
d	0.25	0.25	0.25	0.25

and that $B' = B$. If, instead, we choose $A' = \{(1.00/a)\}$ then also we find that $B' = B$.

In similarity based reasoning we consider the statements

$$p : \text{if } X \text{ is } A \text{ then } Y \text{ is } B, \tau \text{ and } q : X \text{ is } A'.$$

Here A and A' are fuzzy sets defined over the same universe of discourse U and the fuzzy set B is defined over the universe of discourse V , τ is a pure number representing the firing strength of the rule. Let $S(A, A')$ denote the similarity between A and A' (computed using any of the existing definitions). Now, if $S(A, A') > \tau$ then the rule will be fired, i.e., the consequent of the rule is modified to produce the conclusion B' . As an illustration, let $A = \{(1.0, a), (0.75, b), (0.5, c), (0.25, d)\}$, $B = \{(1.0, a), (0.75, b), (0.5, c), (0.25, d)\}$ and $A' = \{(1.0, a), (0.5625, b), (0.25, c), (0.0625, d)\}$. Let us choose τ in such a way that $S(A, A') > \tau$ so that the rule will be fired. In fact, if we choose the Sup-min definition of similarity [13] we find that $S(A, A') = 1.0$. The conclusion in this case is

$$Y \text{ is } B' = \{(1.00, u), (0.75, v), (0.50, w), (0.25, d)\}.$$

Now, if we consider, a totally different rule and a different observation as in the following

$$p : \text{if } X \text{ is } A' \text{ then } Y \text{ is } B, \tau \text{ and } q : X \text{ is } A$$

the conclusion would be the same as earlier.

These methods, proposed in [30,31] use the similarity measure for a direct computation of inference without forming the induced fuzzy relation. Consequently, these methods provide the same conclusion if A and A' are interchanged in the propositions concerned. This is not convincing.

In [11] the authors showed that the notion of membership is a gradual property of fuzzy sets, considered a fuzzy equivalence relation to describe the indistinguishability or similarity in fuzzy sets. Similarity is an important concept for which a crisp model is often found inadequate. There they showed how a crisp set induces a fuzzy set as its extensional hull with respect to a fuzzy equivalence relation. Assuming the similarity modelled by a fuzzy equivalence relation as the basis, fuzzy sets were viewed as induced concepts. Two elements cannot be distinguished by a fuzzy set if they are both either elements of the same set or its complement. They have shown how membership functions of fuzzy sets can be calculated from the fuzzy equivalence relation.

We proposed in a recent paper [29] that a reasoning system should consider every change in A and A' (i.e., the antecedent of the conditional statement and a prototype of the same appearing in the fact) so that the inference is influenced by the said change—more the change (in the linguistic descriptions), less specific is the conclusion. It is also necessary that, nothing better than what the condition reveals should be allowed as a valid consequence. Some form of matching should also be considered in the process of derivation of a consequence, when the fact is different from the rule antecedent.

Considering the fact that similarity is inherent in approximate reasoning, in this paper we attempt to compute the similarity between fuzzy sets and use it in the reasoning mechanism in such a way that change in input is always reflected in the output. Our method of inference is based on a similarity measure. First, the conditional statement is expressed as a fuzzy relation. We interpret it as a conditional fuzzy relation. Then, the similarity between the fact and a prototype of the same appearing in the conditional statement is computed and is used to modify the conditional relation. Such a modification of relation may be performed in many ways. We interpret the modified relation as a fuzzy relation induced by the fact and project the induced relation on the domain of the linguistic variable defining the consequence. In the end, we show the effectiveness of the proposed method to design a rule-based fuzzy system for pattern classification.

The paper is organized in six sections. Section 2 is devoted to similarity measures. In this section we present different similarity measures and study their properties. A set of axioms is proposed to get a reasonable measure of similarity between two fuzzy sets, a model of indistinguishability. New similarity measures are proposed. In Section 3 we present similarity based approximate reasoning that elucidates the connection between similarity and existing approaches to inference in approximate reasoning methodology. For inference in approximate reasoning we use a measure of similarity between the fact(s) and the antecedent of a rule to modify the relation between the antecedent and the consequent of a rule. Finally, a conclusion is derived using the well known projection operation on the domain of the consequent. In the process, Zadeh's compositional rule of inference and existing similarity based reasoning are considered. Section 4 is devoted to the design of fuzzy control based on similarity. A new defuzzification scheme based on the specificity measure of fuzzy sets is proposed. Simulation results are presented for the well-known inverted pendulum problem. Some of our conclusions are presented in Section 5 followed by a list of references in the last section.

2. Similarity indices

In this section, a few measures of similarity between fuzzy sets based on a set of desirable properties are proposed and their properties are analysed. The similarity between two objects suggests the degree to which properties of one may be inferred from those of the other. Providing a measure of similarity depends mostly on the perceptions of different observers. Emphasis should also be given to different members of the set, so that no one member can influence the ultimate result. Many measures of similarity have been proposed in the existing literature [24–28]. A careful analysis of the different similarity measures reveals that it is practically impossible to single out one particular similarity measure that works well for all purpose.

Suppose U be an arbitrary (finite) set, and $\mathcal{F}(U)$ be the collection of all fuzzy subsets of U . Suppose $A, B \in \mathcal{F}(U)$, and a similarity index between the pair $\{A, B\}$ is denoted as $S(A, B; U)$ or simply $S(A, B)$ (can also be considered as a function $S: \mathcal{F}(U)^2 \rightarrow [0, 1]$). We now consider some existing similarity measures.

In certain cases, a measure of difference have been defined first, and then the corresponding similarity measure, viz., in the work of Dubois and Prade [9]. They proposed that $S(A, B)$ should satisfy the following properties:

$$S(A, B) = 1 \quad \text{if and only if} \quad A \nabla B = \emptyset \quad \text{where} \quad \mu_{A \nabla B}(x) = d(\mu_A(x), \mu_B(x)) \quad (3)$$

and

$$d(a, b) = \max[\min(1 - a, b), \min(a, 1 - b)], \quad 0 \leq a, b \leq 1. \quad (4)$$

If A and B have disjoint support then $S(A, B) = 0$. $S(A, B)$ depends on a scalar evaluation of $\overline{A \nabla B}$.

A set theoretic approach may be given by,

$$S(A, B) = \theta f(A \cap B) - \alpha f(A - B) - \beta f(B - A), \quad (5)$$

for some function f satisfying certain characteristics and some parameters $\theta, \alpha, \beta \geq 0$ [33]. Typically, the function f may be taken to be the cardinality function.

Distance functions may also be used to define measures of similarity between two points. The most important class of distance function in this respect is the Minkowski's r -metric, defined as follows:

$$d_r(u, v) = \left[\sum_{i=1}^n |u_i - v_i|^r \right]^{1/r}, \quad r \geq 1, \quad (6)$$

where u and v are two points in an n -dimensional space with components (u_i, v_i) , $i = 1, 2, \dots, n$. Three distance functions, corresponding to $r = 1, 2$ and ∞ , are mostly used [43]. A similarity measure may be defined from the distance functions according to the following:

$$s(\bullet, \bullet) = 1 - d(\bullet, \bullet).$$

Some similarity measures using the Euclidean distance function are as follows:

$$S(A, B) = 1 - \max_{u \in U} (|\mu_A(u) - \mu_B(u)|) \quad (\text{see [22]}), \quad (7)$$

$$S(A, B) = 1 - \frac{\sum_{u \in U} |(\mu_A(u) - \mu_B(u))|}{\sum_{u \in U} (\mu_A(u) + \mu_B(u))} \quad (\text{see [22]}), \quad (8)$$

$$S(A, B) = \frac{1}{n} \sum_{i=1}^n [1 - |\mu_A(u) - \mu_B(u)|], \quad (9)$$

$$S(A, B) = 1 - \sup_{u \in U} |\mu_A(u) - \mu_B(u)| \quad (\text{see [10]}). \quad (10)$$

Several other types of similarity measures can also be found in the existing literature, viz.

$$S(A, B) = \frac{\sum_{u \in U} \{\mu_A(u) \cdot \mu_B(u)\}}{\max\{\sum_{u \in U} \mu_A^2(u), \sum_{u \in U} \mu_B^2(u)\}}, \quad (11)$$

$$S(A, B) = \frac{\sum_{u \in U} \min(\mu_A(u), \mu_B(u))}{\sum_{u \in U} \max(\mu_A(u), \mu_B(u))} \quad (\text{see [22]}), \quad (12)$$

$$S(A, B) = \frac{1}{n} \sum_{i=1}^n \frac{\min(\mu_A(u), \mu_B(u))}{\max(\mu_A(u), \mu_B(u))} \quad (\text{see [34]}), \quad (13)$$

where n , a finite number, is supposed to be the cardinality of the universal set U .

$$S(A, B) = \max_{u \in U} \{ \min(\mu_A(u), \mu_B(u)) \} \quad (\text{see [13]}), \quad (14)$$

$$S(A, B) = \frac{C(A, B)}{\sqrt{T(A) \cdot T(B)}} \quad (\text{see [34]}), \quad (15)$$

where

$$T(A) = \sum_{i=1}^n [(\mu_A^2(x_i) \cdot v_A^2(x_i))]; \quad v_A(x_i) = 1 - \mu_A(x_i), \quad (16)$$

$$C(A, B) = \sum_{i=1}^n [\mu_A(x_i) \cdot \mu_B(x_i) + v_A(x_i) \cdot v_B(x_i)]. \quad (17)$$

It is claimed that such measures of likeness of fuzzy sets may be useful in fuzzy decision making problems.

A family of measures of similarity of fuzzy sets having a strong logical background may be given by

$$S(A, B) = (1/2)[(A \leftrightarrow B) + (\bar{A} \leftrightarrow \bar{B})] \quad (\text{see [23]}), \quad (18)$$

where

$$(A \leftrightarrow B) = (A \rightarrow B) \wedge (B \rightarrow A), \quad (19)$$

\wedge being a conjunction operator and \rightarrow , an implication operator. Different interpretation of the operators will result in different measures of similarity between fuzzy sets.

A simple modification of the last measure gives another measure of similarity of fuzzy sets with finite support as follows:

$$S(A, B) = \frac{1}{n} \sum_{i=1}^n (1/2)[(\mu_A(u) \leftrightarrow \mu_B(u)) + (\mu_{\bar{A}}(u) \leftrightarrow \mu_{\bar{B}}(u))] \quad (20)$$

where

$$(\mu_A(u) \leftrightarrow \mu_B(u)) = (\mu_A(u) \rightarrow \mu_B(u)) \wedge (\mu_B(u) \rightarrow \mu_A(u)), \quad (21)$$

n being the cardinality of the universal set U , \wedge and \rightarrow are as defined earlier. The authors in [2] used this measure to model bi-directional approximate reasoning through an inference network.

Authors in [34,43] have reviewed different similarity measures. It can be shown that all similarity measures listed above satisfy reflexivity, symmetry and boundedness. These three properties can be regarded as necessary for any similarity measure. In this regard, all measures are equally useful. Besides similarity measures should also satisfy properties like computational simplicity and monotonicity.

In line with the works of Tversky [33] on similarities, the authors in [3] presented three measures of similitude, viz., measures of satisfiability, inclusion and resemblance as a proposal for classifying measures of comparison of objects according to their properties. The measures of satisfiability and inclusion do not have the symmetry property, whereas the measure of resemblance satisfies all the three properties. Similitude measures could be used in deductive reasoning to evaluate the extent to which an observation satisfies a given rule.

Similarity measures based on the computation of overall sup/inf-operation between elements are such that they give more importance to a particular value and ignore the presence of others. Thus, two fuzzy sets are often found to be similar when they have the same sup and/or inf value but differ elsewhere. This is not desirable.

Two crisp sets A and B are completely dissimilar only when $A \cap B = \emptyset$. If $A \cap B \neq \emptyset$, then they have some similarity as A and B have some elements in common. The similarity between the two increases as the number of

elements by which the two sets differ decreases. The similarity becomes maximum (the maximum value may be thought of as 1) when the two sets are identical, i.e.,

$$A \cap B = A = B. \quad (22)$$

Here, we consider a direct extension of this concept (22) in defining the similarity between fuzzy sets. For two fuzzy sets, it is reasonable to assume, that the similarity index be zero if and only if $\min(\mu_A(u), \mu_B(u)) = 0$ for all $u \in U$ which is sometimes considered as a definition for $A \cap B = \emptyset$.

In order to provide a definition for similarity index, a number of factors must be considered. A primary consideration is that, whatever way we choose to define such an index, it must satisfy the properties already mentioned. Consider $A, B \in \mathcal{F}(U)$. We expect that, a similarity measure $S(A, B)$ should satisfy the following axioms:

- P1. $S(B, A) = S(A, B)$.
- P2. $0 \leq S(A, B) \leq 1$.
- P3. $S(A, B) = 1$ if and only if $A = B$.
- P4. For two fuzzy sets, A, B , simultaneously not null, if $S(A, B) = 0$ then $\min(\mu_A(u), \mu_B(u)) = 0$ for all $u \in U$.
- P5. If either $A \supseteq B \supseteq C$ or $A \subseteq B \subseteq C$ then $S(A, C) \leq \min\{S(A, B), S(B, C)\}$.

A similarity measure between two fuzzy sets satisfying these axioms can also be termed as an f-near-degree defined in [15,16].

For $0 \leq \epsilon \leq 1$, if $S(A, B) \geq \epsilon$, we say that the two fuzzy sets A and B are ϵ -similar. We now propose two measures of similarity, given by the following definitions:

Definition 1.

$$S(A, B) = \frac{\max_{u \in U} \{\mu_A(u) \wedge \mu_B(u)\}}{\max \{\max_{u \in U} (\mu_A(u)), \max_{u \in U} (\mu_B(u))\}}. \quad (23)$$

Definition 2.

$$S(A, B) = 1 - \left(\frac{\sum_u |\mu_A(u) - \mu_B(u)|^q}{n} \right)^{1/q} \quad (24)$$

where n is the cardinality of the universe of discourse and q is the family parameter.

It is easy to see that the similarity measures given by Definitions 1 and 2 satisfy axioms P1, P2, P3, P4 and P5. The measure proposed in Definition 1 is based on the computation of overall supremum and, therefore, practically difficult to use. According to the definition the fuzzy sets A and B denoted by $\{(\mu_A(u), u)\}$ and $\{(\mu_B(u), u)\}$ and explicitly given by $\mu_A(u) = 1; \forall u$ and $\mu_B(u) = 0; \forall u \neq u_0$ and $\mu_B(u_0) = 1$ are similar as $S(A, B) = 1$. Moreover, the measure will be the same as proposed by the authors of [13] if we consider only normal fuzzy sets. This is why we prefer the measure given by Definition 2.

3. Similarity based approximate reasoning

Many fuzzy systems are based on Zadeh's compositional rule of inference [42]. Despite their success in various systems, researchers have indicated certain drawbacks [31] in the mechanism. This motivates the introduction of similarity based reasoning mechanisms as proposed in [4,5,30,31,36,37].

Different approaches to similarity based reasoning are found in the literature. In [11,12], the authors showed how a crisp set induces a fuzzy set with respect to a fuzzy equivalence relation. Thus, assuming the indistinguishability modelled by a fuzzy equivalence relation as a basic concept, fuzzy sets were viewed as induced concepts [7], i.e., membership degrees can be obtained starting from indistinguishability. These works are mainly concerned with the connection between fuzzy sets and indistinguishability. They defined the degree to which two elements of the universe U cannot be distinguished by a collection of fuzzy sets. In the reasoning procedure, the fact (crisp/fuzzy value) is used to find an induced fuzzy set from the given equivalence relation. This induced fuzzy set is then composed in the usual manner to derive a conclusion.

Table 1
Ordinary approximate reasoning

p :	X is A	then	Y is B
q :	X is A'		
r :			Y is B'

In [31,32], authors proposed a similarity based method called approximate analogical reasoning schema. It was shown that the method is applicable to both point-valued and interval valued-fuzzy sets. In [4], the author proposed two similar methods for medical diagnosis problems. Two other methods based on different modification procedures have been proposed in [36]. In the framework of existing approaches to similarity based inference methodology, recently, in [37], the authors proposed two other similarity based methods for reasoning and made a comparative study of their similarity based fuzzy reasoning methods.

In all these works we notice that the rule, actually a given condition, is not translated to a fuzzy relation between the antecedent and the consequent. Reasoning is based on the computation of the degree of similarity between the fact and the antecedent of a rule, in a rule-based system without considering the intrinsic relation between the antecedent and the consequent. Here, based on the similarity value between the fact(s) and the antecedent of the rule, the membership value of each element of the consequent of the rule is modified to obtain a conclusion. This is the philosophy underlying existing similarity based reasoning schemes reviewed and reported in this paper, the modification procedure alone is different for different schemes.

In such similarity based reasoning schemes, we see that, from a given fact, the desired conclusion is derived using only a measure of similarity between the fact and the antecedent. In some cases, a threshold value τ is associated with a rule. If the degree of similarity, between the antecedent of the rule and the fact, exceeds τ , then only that rule is assumed to be fired. The conclusion is derived using a modification procedure.

As an illustration, let us consider the two premises as in Table 1. Here A and A' are fuzzy sets defined over the same universe of discourse $U = \{u_1, u_2, \dots, u_m\}$ and B, B' are defined over the universe of discourse $V = \{v_1, v_2, \dots, v_n\}$. Let $S(A, A')$ denote some measure of similarity between two fuzzy sets A, A' . In [31], the authors used a kind of normalized Euclidean distance to define the similarity as

$$S(A, A') = [1 + d_2(A, A')]^{-1}. \quad (25)$$

If $S(A, A') > \tau$ only then the rule is considered to be fired and the consequent of the rule is modified to produce the desired conclusion. Based on the change of membership grade of the consequent, two types of modification procedures may be proposed as in [31]—expansion type inference and reduction type inference.

Let $B' = \{(\mu_{B'}(v_i), v_i)\}$ and $s = f(S(A, A'), \tau)$. Then B' will be given by any one of the following form:

- Expansion form: $\mu_{B'}(v_i) = \min(1, \mu_B(v_i)/s)$;
- Reduction form: $\mu_{B'}(v_i) = (\mu_B(v_i) \cdot s)$.

We propose two similarity based approximate reasoning schemes. Our first scheme is a modification of the scheme presented in [31]. With different results we show that the proposed similarity based approximate reasoning methods are reasonable. In the proposed methods, for inference in a rule-based system, the conditional rule is first expressed as a fuzzy relation (translation). To construct the relation, we prefer to use triangular norms for a better understanding. Other interpretations are also possible. In a rule based system, we compute the similarity between the fact and the antecedent of the rule to modify the above fuzzy relation and not the consequence of the rule as is done in existing similarity based reasoning mechanisms (matching). The modification is based on a measure of similarity following some scheme which will be presented in the sequel. The result is interpreted as the induced fuzzy relation (modification). Then the inference is computed from the induced fuzzy relation using the well known projection operation (inference).

The author in [4] proposed that if $S(A, A') \geq \tau$, the predefined threshold value, then the rule will be fired and strength of confirmation is calculated by $S(A, A') \cdot \mu$, where μ is the confidence factor associated with the rule. In [5], the same author used weights with each propositions for the calculation of similarity. In this case, the similarity

between fuzzy sets is computed as:

$$S(A, A') = \sum_{i=1}^m \left[T(\mu_{A'}(u_i), \mu_A(u_i)) \cdot \left(w_i / \sum_{k=1}^m w_k \right) \right]$$

where

$$T(\mu_{A'}(u_i), \mu_A(u_i)) = 1 - |\mu_{A'}(u_i) - \mu_A(u_i)|.$$

The procedure for the computation of the conclusion remains the same.

The authors in [37] used the values of certainty factor associated with the rules in the modification procedure. The inference is based on the number of propositions in the antecedent of the rule(s) as well as the operator(s) connecting them. In each case, the inference is of expansion type. Two other fuzzy reasoning methods are also proposed in [37]. One modification is based on Zadeh's inclusion and cardinality measures and the other is based on equality and cardinality measures. Other operations remain almost identical. In the following, we propose a similarity based approximate reasoning technique considering the drawbacks already mentioned.

3.1. Proposed method

Let X, Y be two linguistic variables and let U, V respectively denote the universes of discourse. Two typical propositions p and q are given and we may derive a conclusion according to similarity based inference scheme as described in Table 1. Let fuzzy sets A, A', B and B' in Table 1 be defined as:

$$\begin{aligned} A &= \{(\mu_A(u_i), u_i)\}; \\ A' &= \{(\mu_{A'}(u_i), u_i)\}; \\ B &= \{(\mu_B(v_i), v_i)\}; \\ B' &= \{(\mu_{B'}(v_i), v_i)\}. \end{aligned}$$

Unlike the existing methods, here we translate the conditional statement into a fuzzy relation. Then the similarity between the fact and the antecedent of the rule is used to modify this relation. Here every change in the concept, as it appears in the conditional premise and in the fact, is incorporated into the induced fuzzy relation. Then a conclusion may be drawn using the projection operation. This conclusion is influenced by the change in the fact and the antecedent of the rule fired.

In order to avoid the use of certainty factor for rule-misfiring, we modify the inference scheme in such a way that a significant change will make the conclusion less specific. This is done by choosing an expansion type of inference scheme. Here, the UNKNOWN case, i.e., the fuzzy set $B' = V$, is to be taken as the limit of non-specificity. Explicitly, when the similarity value becomes low, i.e., when A and A' differ significantly, the inference should be $B' = V$. As $A' = A$, we expect that $B' = B$ and for all other A' , the relation $B' \supseteq B$ holds. This in turn implies that, nothing better than what the rule says should be allowed as a valid conclusion.

In view of the above observations, we propose a scheme for computation in the following algorithm.

Algorithm SAR (Similarity based Approximate Reasoning)

- Step 1 Translate premise p and compute $R(A, B)$ using some suitable translating rule (possibly, a T-norm operator).
- Step 2 Compute $S(A, A')$ using some suitable definition (possibly, Definition 2).
- Step 3 Modify $R(A, B)$ with $S(A, A')$ to obtain the modified conditional relation $R(A, B | A')$ using some scheme C .
- Step 4 Use sup-projection operation on $R(A, B | A')$ to obtain B' as

$$\mu_{B'}(v) = \sup_u \mu_{R(A, B | A')}(u, v). \quad (26)$$

Now, we propose two schemes C1 and C2 for computation of the modified conditional relation $R(A, B | A')$ as given in Step 3.

Scheme C1

The first scheme C1 is based on a concept similar (but not identical) to the scheme proposed in [31]. The authors computed the conclusion $B' = \min(1, B/s)$, where s is the measure of similarity between fuzzy sets A and A' without considering the information suggested by the conditional rule. We modify the conditional relation according to the following

$$R(A, B | A') = [r'_{u,v}]_{m \times n} = \begin{bmatrix} r'_{u,v} = \min(1, r_{u,v}/s) & \text{if } s > 0 \\ = 1 & \text{otherwise.} \end{bmatrix} \quad (27)$$

It is clear that, the proposed scheme, unlike the scheme in [30,31], does not always produce the same conclusion when A and A' are interchanged. In (27), if $s \leq r_{u,v}$ for some $v \in V$ then $r'_{u,v}$ becomes equal to one, thus making its membership value, in the resultant fuzzy set, 1.

This scheme, although a heuristic one, is intuitively a plausible one. Our next scheme for the computation of modified fuzzy relation $R(A, B | A')$ is based on a set of axioms.

Scheme C2

We believe that in a similarity based reasoning methodology, a scheme for computation of the induced relation, when the fact and the conditional statement is given, should satisfy the following axioms:

- AC1. If $S(A, A') = 1$, then $\mu_{R(A, B | A')}(u, v) = \mu_{R(A, B)}(u, v)$;
- AC2. If $S(A, A') = 0$, then $\mu_{R(A, B | A')}(u, v) = 1$;
- AC3. As $S(A, A')$ increase from 0 to 1, $\mu_{R(A, B | A')}(u, v)$ decrease uniformly from 1 to $\mu_{R(A, B)}(u, v)$.

Axiom AC1 asserts that we should not modify the conditional relation as and when A' and A remain equal. Axiom AC2 asserts that when A' is completely dissimilar to A , we should not conclude anything specifically. In such a situation, anything is possible. AC3 says that as the fact A' changes from the most dissimilar case (similarity value zero) to the most similar one (similarity value one), the inferred conclusion should change from the most non-specific case, i.e., the UNKNOWN case ($B' = V$) to the most specific case, i.e., $B' = B$. This, in turn, means that whatever A' be, $R(A, B | A') \geq R(A, B)$, i.e., the induced relation should not be more specific than what is given as a condition.

For notational simplicity, let us denote $S(A, A')$ by s and $R(A, B | A')$ by r' . Now, axiom AC3 suggests a function of the form $\frac{dr'}{ds} = k$ (a constant) $\Rightarrow r' = ks + c$, c is a constant. These two constants may be determined from the conditions already prescribed in axioms AC1 and AC2. More explicitly, when $s = 1$ we know that $r' = r$ (from axiom AC1) and when $s = 0$ we know that $r' = 1$ (from axiom AC2). This gives,

$$r' = 1 - (1 - r).s \quad (28)$$

as our new scheme for the modification of the conditional relational. Therefore, axiom AC1 through axiom AC3 suggest the Scheme C2 as

$$\mu_{R(A, B | A')}(u, v) = 1 - (1 - \mu_{R(A, B)}(u, v)).S(A, A'). \quad (29)$$

The same could be derived from a different interpretation as well. Any transformation of the kind

$$R(A, B | A')(u, v) = I(s, R(A, B)(u, v))$$

where I is an implication function (hence satisfying $I(x, 0) = 1$, $I(1, x) = x$ and decreasing continuously in the first variable) fulfils the postulates AC1, AC2, and AC3. In particular, taking I as the residuated implication of product t -norm (i.e., $I(x, y) = \min(1, y/x)$) we can obtain Scheme C1, and taking I as strong product implication ($I(x, y) = 1 - x + xy$) we can obtain Scheme C2. There are many more functions which can be used to generate the scheme for modification.

From (26) and (27) it is found that when $S(A, A') = 0$ we have $B' = V$, in other words, it is impossible to conclude anything when $\{A, A'\}$ are completely dissimilar. When $S(A, A')$ is close to unity, $R(A, B | A')$ is close to $R(A, B)$

and hence the inferred fuzzy set B' will be close to B , i.e., $S(B, B')$ is close to unity. Axiom AC3 also ensures that a small change in the input produces a small change in the output and hence, in this sense the above mechanism of inference is stable. As in the previous case, we see that in (27), if either $S(A, A') = 0$ or $\mu_{R(A,B)}(u, v) = 1$ then $r'_{u,v}$ becomes equal to one.

Remarks. If Mamdani's min-rule is used for the translation of the implication statement and only normal fuzzy sets are considered in the manipulation then $A' = A$ will imply that $B' = B$. This is simply because, in this case, $S(A, A') = 1$ and hence $R(A, B | A')$ will be equal to $R(A, B)$.

Let A be a normal fuzzy set. If we assume that the translating rule used in generating the conditional relation is one of T-norm type then, as is already proposed, a basic and desirable result of the inferred proposition, 'nothing better than what the rule says may be concluded', may be established as in the following. For that, let us consider the model as in Table 1.

Theorem 1. For all $A, A', B' \supseteq B$.

Proof. Let us first consider Scheme C2. From (26) and the result of application of Scheme C2, we have,

$$\begin{aligned}\mu_{B'}(v) &= \sup_{u \in U} \mu_{R(A,B|A')}(u, v) \\ &= \sup_{u \in U} \{1 - (1 - \mu_{R(A,B)}(u, v)) \cdot S(A, A')\} \\ &\geq \sup_{u \in U} \{\mu_{R(A,B)}(u, v)\}; \quad (\text{since, } 0 \leq S(A, A') \leq 1)\end{aligned}$$

i.e.,

$$\mu_{B'}(v) \geq \sup_{u \in U} \{\mu_A(u) \circ \mu_B(v)\}, \quad \text{where } \circ \text{ is any T-norm operator.}$$

Therefore, $\forall v \in V, \mu_{B'}(v) \geq \mu_B(v)$, since A is normal.

Let us now consider Scheme C1. From (26) and using the result of application of Scheme C1, we have,

$$\begin{aligned}\mu_{B'}(v) &= \sup_{u \in U} \mu_{R(A,B|A')}(u, v) \\ &= \sup_{u \in U} \min\{1, \mu_{R(A,B)}(u, v)/S(A, A')\} \\ &\geq \sup_{u \in U} \{\mu_{R(A,B)}(u, v)/S(A, A')\} \\ &\geq \sup_{u \in U} \mu_{R(A,B)}(u, v); \quad (\text{since, } 0 \leq S(A, A') \leq 1)\end{aligned}$$

i.e.,

$$\mu_{B'}(v) \geq \sup_{u \in U} \{\mu_A(u) \circ \mu_B(v)\}, \quad \text{where } \circ \text{ is any T-norm operator.}$$

Therefore, $\forall v \in V, \mu_{B'}(v) \geq \mu_B(v)$, since A is normal.

Let us observe here that Schemes C1 and C2 can be put as particular cases of a more general form

$$\mu_{R(A,B|A')}(u, v) = (s \rightarrow \mu_{R(A,B)}(u, v))$$

where \rightarrow is any implication function. In particular, when $(x \rightarrow y) = \min(1, y/x)$ we get Scheme C1 and when $(x \rightarrow y) = \min(1 - x + xy)$, we get Scheme C2. \square

Let us now calculate $\mu_{B'}(v)$ using Algorithm SAR for the above-mentioned general form under the totally plausible hypothesis that A is normal (i.e., $\sup_u \mu_A(u) = 1$) and A does not completely cover the domain (i.e. $\inf_u \mu_A(u) = 0$). We have $\mu_{B'}(v) = \sup_u \mu_{R(A,B|A')}(u, v) = \sup_u (s \rightarrow \mu_{R(A,B)}(u, v)) = (s \rightarrow \sup_u \mu_{R(A,B)}(u, v))$. Now we are basically left with two mostly used cases, according to how the implication is modelled in the rules:

Case 1: If $\mu_{R(A,B)}(u, v) = T\mu_A(u), \mu_B(v)$, then

$$\begin{aligned}\mu_{B'}(v) &= (s \rightarrow \sup_u \mu_{R(A,B)}(u, v)) \\ &= (s \rightarrow \sup_u T(\mu_A(u), \mu_B(v))) \\ &= (s \rightarrow T(\sup_u \mu_A(u), \mu_B(v))) \\ &= (s \rightarrow T(1, \mu_B(v))) \\ &= (s \rightarrow \mu_B(v)).\end{aligned}$$

This derivation is completely general for any implication \rightarrow . Scheme C1 corresponds to taking $x \rightarrow y = \min(1, y/x)$ and this actually leads to

$$\mu_{B'}(v) = \min(1, \mu_B(v)/s)$$

which is exactly Turksen's expansion form. This is not the case for subnormal fuzzy set A for which our case is a more general one. On the other hand, Scheme C2 corresponds to taking $(x \rightarrow y) = 1 - x + xy$, leading to

$$\mu_{B'}(v) = 1 - s + s \cdot \mu_B(v)$$

that corresponds to normalizing Turksen's reduction scheme by adding a constant value $1 - s$ to all elements of the domain. This shows that Turksen's scheme of similarity based reasoning is a derivative of the proposed similarity based approximate reasoning scheme.

Case 2: If $\mu_{R(A,B)}(u, v) = I(\mu_A(u), \mu_B(v))$, then

$$\begin{aligned}\mu_{B'}(v) &= (s \rightarrow \sup_u \mu_{R(A,B)}(u, v)) \\ &= (s \rightarrow \sup_u I(\mu_A(u), \mu_B(v))) \\ &= (s \rightarrow I(\inf_u \mu_A(u), \mu_B(v))) \\ &= (s \rightarrow I(0, \mu_B(v))) \\ &= (s \rightarrow 1) = 1.\end{aligned}$$

This always leads to the conclusion that B' could be anything. Therefore Algorithm SAR under Case 2 is meaningless. This justifies our preference over the use of T-norms in translation.

Now, a rule-base hardly contains rules with only one clause in the antecedent. For rule-base with multiple clauses in the antecedent, we may apply the proposed scheme as in the following:

Let X_1, X_2, \dots, X_k, Y be $k + 1$ -linguistic variables defined respectively over universes of discourse U_1, U_2, \dots, U_k, V and let $U_i = \{u_i^j\}; j = 1, 2, \dots, j_i$. Let us consider an application of the basic reasoning scheme to rules with multiple clauses in the antecedent as presented in Table 2.

The problem is to find the linguistic value of the variable Y as suggested by the rules, when the values of the k -antecedent variables are given [20].

Table 2
Applicable form of approximate reasoning

if	X_1 is A_{11}	and	X_2 is A_{12}	...	X_k is A_{1k}	then	Y is B_1
else if	X_1 is A_{21}	and	X_2 is A_{22}	...	X_k is A_{2k}	then	Y is B_2
	\vdots		\vdots		\vdots		\vdots
else if	X_1 is A_{m1}	and	X_2 is A_{m2}	...	X_k is A_{mk}	then	Y is B_m
	X_1 is A_1	and	X_2 is A_2	...	X_k is A_k		
Conclusion							Y is B

Under the conventional mechanism, for each rule, the consequent fuzzy set is calculated according to existing method of inference as already described and then the union of all consequent fuzzy sets is taken as the conclusion which is then defuzzified, if necessary, using some defuzzification scheme.

In the present case of similarity based reasoning such calculation is not possible as the membership values computed from the modified induced relation becomes less and less specific as the similarity between the facts and antecedent of a rule decreases. The membership values of various elements become equal to the maximum, making it an ambiguous one (more alternatives with similar membership values at the highest level). For example, in case of Mamdani-type of reasoning [1,17–19], if the firing strength of a rule is, say 0.3, then all alternatives which have membership values greater than or equal to 0.3 take membership values equal to 0.3. If we choose Scheme C1 in the present case, and if the similarity value is 0.3, then the membership values of elements in the inferred fuzzy set will be at least 0.3. Moreover, the elements having membership value greater than or equal to 0.3 in the consequent of the rule will be equal to ‘1’ in the consequent fuzzy set. This means that, with decrease in similarity the computed membership values increase and ultimately move close to the least specific case (with membership values of 1 for all alternatives).

It is because of this that we propose a new scheme for a conclusion, based on a measure of similarity. Our method is based on rule-selection and then rule-execution. In both cases, we use the concept of similarity between fuzzy sets as a basis of the task. For that, first of all, we compute $S(A_j, A_{ij})$; $j = 1, 2, \dots, k$ and perform the same operation for different $i = 1, 2, \dots, m$. Let s_{ij} denote the different similarity values. Next, we compute the overall rule matching index from the above data as

$$s^i = \min_j s_{ij}. \quad (30)$$

From among the m distinct rules, we choose those rules for which $s^i > \epsilon$, ϵ may be interpreted as a threshold in our case. Then we apply Algorithm SAR to generate a conclusion from each rule conformal for firing. The overall output may be generated using the intersection of conclusions (fuzzy sets) resulted from different fired rules. It is important to note here that the intersection operation is chosen in order to justify the rule-selection procedure. Here, fewer rules are fired and the output of each rule is significant. The algorithm may be schematized as follows:

Algorithm ESAR (*Extended Similarity based Approximate Reasoning*)

Step 1 For $i = 1, 2, \dots, m$

compute s_{ij} ; $j = 1, 2, \dots, k$ and then
set $s^i = \min_j s_{ij}$.

Step 2 Set ϵ and then find rules for which $s^i > \epsilon$.

Step 3 Translate the i th-rule as obtained in Step 2 and compute the relation R_i using any suitable translating rule possibly, a T-norm operator.

Step 4 Modify R_i with s^i to obtain the modified relation R'_i according to either (27) or (29).

Step 5 Use sup-projection operation on R'_i to obtain B'_i .

Step 6 Perform Steps 3 to 5 for all i for which $s^i > \epsilon$. Compute the output $B = \bigcap_i B'_i$.

4. Fuzzy control

In this section we attempt to design rule-based fuzzy systems using the concept of similarity between fuzzy sets and similarity based approximate reasoning instead of conventional approximate reasoning. Approximate reasoning, the inference mechanism, plays a leading role in many areas of application. Many scientific problems in real-world setting may be solved using approximate reasoning methodology. We present a rule-based solution to fuzzy control. Its application in pattern classification has been considered in [29].

Zadeh outlined the basic ideas underlying fuzzy control in [39,41]. Among them, the concept of linguistic variables, fuzzy if-then rules, fuzzy algorithms and the compositional rule of inference are some essential ideas in the design of a fuzzy controller. However, it was the seminal work of Mamdani and Assilian [18,19] that showed how these

ideas could be translated into a working control system. Since then fuzzy logic control has attracted great attention from both the academic and industrial communities. Considerable progress has been made in applying fuzzy logic control successfully to industries. Fuzzy logic control technology has drastically reduced the development time and deployment cost for the synthesis of non-linear controllers for dynamic systems. All these gave fuzzy logic a much higher visibility in control.

Rule-based fuzzy control is a useful tool for systems where the exact model is either not known or is too complex to be tractable in real time. The design of a rule-based fuzzy controller consists of a sequence of activities such as knowledge acquisition, controller structure definition, selection of rules, tuning a variety of gains and other controller parameters, modification of rules to improve performance and defuzzification.

Fuzzy set theory may be applied to the field of fuzzy logic control (FLC). Fuzzy controllers may perform better in some cases than conventional model-based controllers, especially, when applied to processes difficult to model; and when there is a significant heuristic knowledge from human operators available. Important properties of fuzzy controllers are their high flexibility and low sensitivity to parametric variations, which enables their application to varying problems.

The action of a fuzzy controller is based on a collection of *if-then* rules whose antecedent(s) and consequent are fuzzy values. The process state parameters are first fuzzified and then combined with these rules for a fuzzy action. The fuzzy action is then defuzzified for a precise action for the said process. Here, we use similarity concept in deriving the control action. In order to demonstrate the effectiveness of similarity based approach to inference, we use a new specificity based defuzzification procedure. To do that, let us begin with a discussion on some basic concepts underlying conventional fuzzy control.

4.1. Conventional fuzzy control

To simplify the process we consider a multi-input single output (MISO) fuzzy system. A block diagram of such a fuzzy logic controller is shown in Fig. 1.

In Fig. 1, the physical values of the current process state variables i.e., the error and the change of error are respectively denoted as e and Δe . The two symbols y and u respectively denote the system output and the controller output. The process state variables (e and Δe) are sometimes mapped onto normalized domain by the input scaling factors G_e and $G_{\Delta e}$. This stage is optional. When G_e and $G_{\Delta e}$ are used, the controller output variable (u) is mapped onto its physical domain by the output scaling factor G_u . This is required only when the rule-base produce an output over a normalized domain.

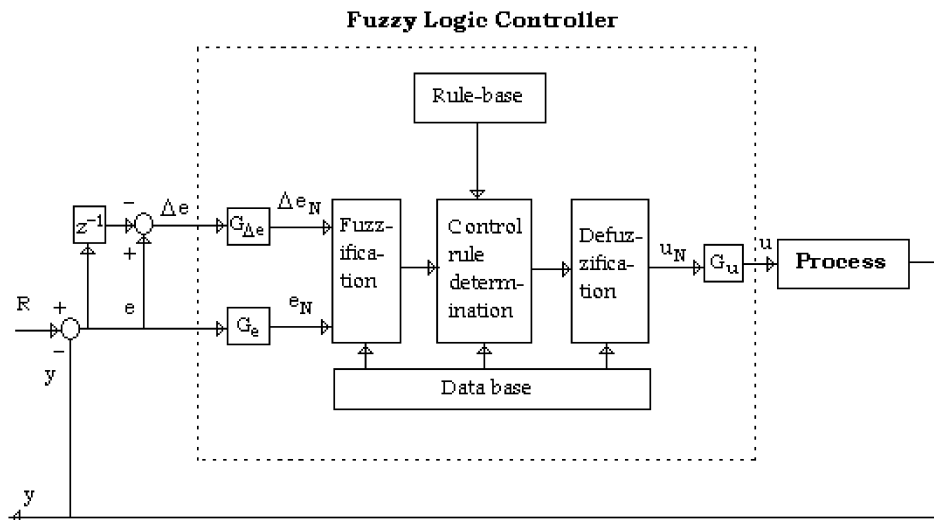


Fig. 1. Block diagram of a fuzzy logic controller.

The input–output behaviour of a MISO system may be modelled as a rule-based fuzzy system consisting of a set of fuzzy *if–then* rules as defined below:

If X_i is positive small and X_j is near zero then U is positive large.

In general, the i th-rule may be represented in a standard form as

If X_1 is A_1^i and X_2 is A_2^i and ... X_n is A_n^i then Y is B_i

where X_j ; $j = 1, 2, \dots, n$ are n linguistic variables defining the process states and A_j^i ; $j = 1, 2, \dots, n$ are the linguistic descriptions of the respective variables, appearing in the antecedent part of i th-rule, and B_i , the rule consequence, is a linguistic description of the variable Y , defining the fuzzy control action.

Fuzzy model with rules of the above form is popularly known as Mamdani–Assilian (MA) type model. A different model, also in use, is known as Takagi–Sugeno (TS) model where the rules are of the form:

If X_1 is A_1^i and X_2 is A_2^i and ... X_n is A_n^i then $u = u(X_1, X_2, \dots, X_N)$.

In this paper we consider only the MA-type model. The computational aspects of such a fuzzy system involves the following basic steps:

Fuzzification The inputs to this stage are the actual point-wise real values (normalized) of the process state variables. The respective real values of the process state variables are converted to fuzzy subsets of membership values defined over the domain of the concerned variable. For each state variable, the data base contains relevant information necessary for the said fuzzification. The output of this stage are the fuzzy set representation of error (e) and change of error (Δe).

Knowledge base The knowledge base of a fuzzy system consists of a data base and a rule-base. The basic function of the data base is to provide necessary information for proper functioning of the fuzzification module, the rule-base and the defuzzification module. Whereas, function of the rule-base is to represent, in a structured way, the control policy of an experienced process operator and/or control engineer given in the form of a set of rules as described earlier.

Inference This is a decision making stage under the control rule determination block, which generates fuzzy output corresponding to fuzzified inputs using fuzzy rules from the rule-base. A typical fuzzy rule actually enumerate the process input–output relation on a portion of the input space. The method of approximate reasoning is applied in the derivation of the fuzzy output. The inputs to this block are the fuzzified state variables, which are first used in rule selection and then used in output determination through rule firing. Two types of inference are commonly used in the design of fuzzy logic controllers with multiple rules. In the first case, all rules in the rule-base are combined to produce a single relation to represent the input–output behaviour of the system over the entire space. Then the fuzzy input is used to infer according to compositional rule of inference. The resultant is a fuzzy set. In the second case, each rule is fired separately with the same fuzzy input, using approximate reasoning mechanism, to generate the fuzzy output of the rule and then the resulting fuzzy outputs are combined to produce a single fuzzy set as the possible fuzzy action.

Defuzzification The result of rule firing, using any of the above mentioned approaches to inference, is a fuzzy set. This is interpreted at the semantic level as the possible values of the desired output. We need to determine a precise action for the process to be controlled. The purpose of defuzzification is to obtain a scaler value $u \in U$, from the said output fuzzy set, as the control action. Then, if necessary, de-normalization is performed on the output so as to obtain the corresponding action on its physical domain.

Different methods of defuzzification are in use [6]. Let A be any fuzzy set defined over the universe of discourse $U = \{u_1, u_2, \dots, u_n\}$ and let a defuzzified value of A be denoted as u^* . In the following, some of the well known methods of defuzzification for the computation of u^* are presented.

Center-of-gravity

This is the most commonly used defuzzification method. Here, the defuzzified value is given by

$$u^* = \frac{\sum_{i=1}^n u_i \cdot \mu_A(u_i)}{\sum_{i=1}^n \mu_A(u_i)}. \quad (31)$$

First-of-maxima

This is the simplest of all the defuzzification schema, in use. As the name suggests, first member with the maximum membership value in the output fuzzy set is taken as the corresponding defuzzified value. Thus, the defuzzified value will be given by

$$u^* = \inf_{u \in U} \left\{ u \in U \mid \mu_A(u) = \max_{v \in U} \mu_A(v) \right\}. \quad (32)$$

We may use an alternative version of the above scheme, known as the last-of-maxima.

So far, we are concerned with defuzzification of a fuzzy set. In rule-based fuzzy control, when we use the firing of individual rules for inference, we find a collection of clipped fuzzy sets as possible outputs. Let us now consider some widely used methods of defuzzification procedure in such cases. For that, let there be m fuzzy sets $\{A^{(k)}; k = 1, 2, \dots, m\}$.

Center-of-sums

In this method, the overlapping areas are considered more than once. The computational part is simple as compared with Center-of-gravity, and works fast. Here, the defuzzified output value is given by

$$u^* = \frac{\sum_{k=1}^m \sum_{i=1}^n u_i \cdot \mu_A^{(k)}(u_i)}{\sum_{k=1}^m \sum_{i=1}^n \mu_A^{(k)}(u_i)}. \quad (33)$$

It is to be noted that the Center-of-gravity method of defuzzification for clipped fuzzy sets is exactly the same as Center-of-sums method of defuzzification, except that the latter uses the overlapped areas only once.

Height

This method of defuzzification demands strictly convex fuzzy sets. The individual peak values of consequent fuzzy sets of the fired rules are used to generate the weighted average of these peak values. It is a simple method and works faster than the Center-of-sums method. Let $p^{(k)}$ be the peak value of $A^{(k)}$ and $h^{(k)}$ be the corresponding height of the clipped version of $A^{(k)}$ or the firing strength of the k th rule. The defuzzified value will then be given by

$$u^* = \frac{\sum_{k=1}^m p^{(k)} \cdot h^{(k)}}{\sum_{k=1}^m h^{(k)}}. \quad (34)$$

A detailed study on the conventional approach to fuzzy control may be found in [14].

4.2. Similarity based fuzzy control

Let us now propose a different strategy for fuzzy control based on the concept of similarity. The concept of similarity between fuzzy sets are used in selecting rules from the rule-base, to be fired for the particular input specification and then in deriving the control action from a set of rules and input values. In this regard, a new scheme for defuzzification based on a measure of specificity of fuzzy sets is proposed. Let us consider a process controlled by p inputs.

Fuzzification Since we are considering similarity based reasoning, in this module, we propose to consider only parametric, functional definitions of certain uniform geometric shaped fuzzy sets. The most popular choices include, triangular, trapezoidal and bell-shaped functions. Among them, the parametric definition of a triangular shaped function is the most economic one. We use a triangular membership function both for fuzzification and rule-base generation. Since, we are considering similarity concept in choosing a rule from the rule-base for possible firing, the real values of each process state variables are fuzzified using similar triangles, i.e., of the same width and height as is used in the rule-base. As for example, let e_t denote the observed value of the state variable error e at time t , then e_t is fuzzified by a symmetric triangular membership function with peak at e_t (membership grade 1) and base-width $b_t = b$, where b is the base-width of every fuzzy set defined for the linguistic variable corresponding to error, for the purpose of rule-base generation. The values of b_t may be different also. Note here that, in conventional fuzzy controller design, normally fuzzy singletons are used.

Knowledge base This module remains the same as described in the previous section.

Inference We shall consider, similarity concept, only rule-based inference here. We first select the rules to be fired for a given input combination. In selection, we use the similarity between the given and the antecedent fuzzy sets as a basis. In order to select the rules for firing, we proceed as follows: Let A_i' be the fuzzified values of x_i ; $i = 1, 2, \dots, p$. x_i is the i th-component of the state vector $\mathbf{x} = (x_1, x_2, \dots, x_p) \in R^p$. Let the k th rule be

R^k : if x_1 is A_1^k and x_2 is A_2^k and \dots and x_p is A_p^k then u is U^k .

Let us set

$$\alpha_k = \min_{i=1,2,\dots,p} \{S(A_i', A_i^k)\}. \quad (35)$$

Now, R^k is fired if $\alpha_k > 0$. This value of α_k may be interpreted, at the semantic level, as a measure of agreement between the fact and the antecedent of the k th rule. This completes the first part of reasoning.

In the rule execution stage of inference, we first compute the input–output relation from the translation of the rule in consideration. Then we modify the said relation using the rule firing strength (α_k) and compute the output fuzzy action.

Defuzzification The result of rule firing is a class of clipped fuzzy sets defined over the same universe of discourse. We are required to determine a single real value from those fuzzy outputs. Earlier we discussed several methods of defuzzification like Center-of-gravity, Center-of-sums, height, etc. For these schema, the basic idea is as follows: If the membership grade of a particular element, in the output fuzzy set, is high then this contributes more to the defuzzified output.

Such concepts cannot be used in the present case of similarity based reasoning paradigm. Here, the lower the similarity value between the rule-antecedent and the fact, the closer the output to the least specific case (i.e., unknown) with the membership grade of elements in the output fuzzy set close to ‘1’. In such cases, a natural choice would be to use specificity information of the output fuzzy sets in defuzzification. In our scheme, the basic idea will be: the element with high membership value should come from the most specific output fuzzy set.

Our first defuzzification scheme is based on this concept. The most specific among the output fuzzy sets has the maximum impact on the resultant choice. For that we compute specificity value of each output fuzzy set separately.

Specificity based defuzzification

Let there be m clipped fuzzy sets $\{A^{(k)}; k = 1, 2, \dots, m\}$ and let $\{s^{(k)}, p^{(k)}; k = 1, 2, \dots, m\}$ be the specificity [8,35] associated with $A^{(k)}$ as well as the height of the consequent of the k th-rule. Then the defuzzified value u^* will be given by

$$u^* = \frac{\sum_{k=1}^m p^{(k)} \cdot s^{(k)}}{\sum_{k=1}^m s^{(k)}}. \quad (36)$$

Other methods of defuzzification may also be defined. We next suggest another scheme which looks similar to the center-of-gravity method as described earlier. Among the set of rules fired, the rule corresponding to the best match between the fact and the antecedent will give the most specific output as compared to the outputs from other rules having lower similarity value. Thus, unlike conventional defuzzification techniques, the conjunction of all output fuzzy sets may be taken as the output of the rule-based system, which then may be defuzzified using center-of-gravity method of defuzzification as already defined.

Modified center-of-gravity

Let there be m clipped fuzzy sets $\{A^{(k)}; k = 1, 2, \dots, m\}$ and let $A = \bigcap_{i=1}^m A^{(k)}$. Then the defuzzified value u^* will be given by

$$u^* = \frac{\sum_{i=1}^n u_i \cdot \mu_A(u_i)}{\sum_{i=1}^n \mu_A(u_i)}. \quad (37)$$

Based on above discussions, an algorithm may be schematized as follows:

Algorithm FC (Fuzzy Control)

- Step 1* Let $\mathbf{x}^t = (x_1^t, x_2^t, \dots, x_p^t)$ be the process state vector at time t .
- Step 2* Fuzzify each x_i^t , the real value for the process state variable, using triangular membership function.
- Step 3* For each rule R^k , compute similarity index between the input fuzzy set and the antecedent fuzzy set, for all variables. Obtain α^k , the minimum of all the similarity indices. Take this as a matching grade of the rule.
- Step 4* Perform similarity based approximate reasoning, taking one rule at a time for which $\alpha^k \geq \epsilon$; $0 \leq \epsilon \leq 1$.
- Step 5* Defuzzify the fuzzy sets, as obtained in Step 4, by using either the specificity based defuzzification scheme or the modified center-of-gravity method.
- Step 6* Use the defuzzified result, as found in Step 5, as the process input for the next time interval. Set $t \leftarrow t + 1$. Go to Step 2.

4.3. Inverted Pendulum problem

In this section, we consider an illustrative application of the proposed algorithm *Fuzzy Control* for controlling an inverted pendulum mounted on a cart as may be seen in Fig. 2.

The problem is to balance the pole in a vertical position by applying an appropriate horizontal force u on the cart. Here, we shall consider only the motion of the angular position of the pole on the cart, and not the position of the cart or the velocity of the cart. Let us assume that the cart travels in one direction only along a frictionless track, i.e., the motion to be purely two-dimensional. It is also assumed that the pendulum mass is concentrated at the end of the rod and the rod is massless. The inverted pendulum is unstable in a sense that it may fall over anytime in any direction unless a suitable control force is applied. Therefore, the problem is to design a control system in a such way that, given any initial conditions (may be caused by disturbances), the pendulum may be brought back in the vertical position. The motion of the cart will not be considered. A detailed description may be found in [21].

For a solution to the above problem, let us first define the process parameters. Let θ be the angular position of the pole with respect to the vertical. Since we expect to keep the inverted pendulum vertical, the angle θ is assumed to be small. Let u be the driving force on the cart, $2l$ be the length of the pole, g be the gravitational acceleration (a constant), M be the mass of the cart and m be that of the pole.

Let x_1, x_2, x_3 and x_4 denote the four state variables used to represent the angle dynamics of the pole as under

$$x_1 = \theta,$$

$$x_2 = \dot{\theta},$$

$$x_3 = x,$$

$$x_4 = \dot{x},$$

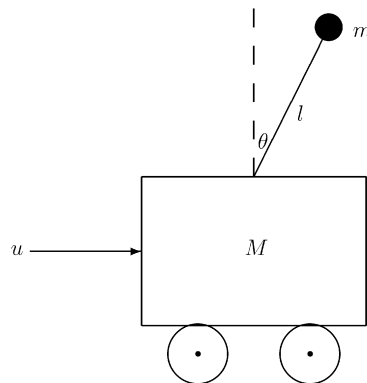


Fig. 2. The Inverted Pendulum on a moving cart.

where x denotes the location of the cart from a fixed point, about which the pendulum moves. It is easy to describe the dynamics of the motion of the pendulum [21] in terms of state equations as below:

$$\dot{x}_1 = x_2, \quad (38)$$

$$\dot{x}_2 = \frac{M+m}{Ml}gx_1 - \frac{1}{Ml}u, \quad (39)$$

$$\dot{x}_3 = x_4, \quad (40)$$

$$\dot{x}_4 = -\frac{m}{M}gx_1 + \frac{1}{M}u, \quad (41)$$

where g is taken to be 9.8 m/s^2 . Let us set $m = 0.1 \text{ kg}$, $M = 2 \text{ kg}$ and $l = 0.5 \text{ m}$ as exactly in [21]. These numerical values when substituted in the set of Eqs. (37)–(40), we obtain

$$\dot{x}_1 = x_2, \quad (42)$$

$$\dot{x}_2 = 20.601x_1 - u, \quad (43)$$

$$\dot{x}_3 = x_4, \quad (44)$$

$$\dot{x}_4 = -0.4905x_1 + 0.5u. \quad (45)$$

In this example, the operating range for θ (the angle) is set to be $[-0.1, 0.1]$, measured in radian and that for $\dot{\theta}$ (the angular velocity) is set to be $[-0.15, 0.15]$, measured in radian/sec. Over these ranges, seven equi-spaced (similar) as well as half-overlapping (crosses at the membership value 0.5) isosceles triangular fuzzy sets have been generated off-line, in order to define seven linguistic labels for each of the linguistic variables. Using trial and error method, the operating range for the applied force is found to be $[-6.0, 6.0]$, measured in Newton unit. Here seven similar such fuzzy sets are generated off-line. According to a human expert's description the following fuzzy conditional statements may be taken to represent the relation between the input (the angle and the angular velocity) and the output (the controlling force to be applied on the cart) variables. The rule-base is shown in Table 5. There are seven rows and an equal number of columns in the table, indicating the presence of exactly forty-nine distinct rules in the rule-base. Considering the symmetry of the motion of the pendulum about the vertical, the relational Table 5 is found to be symmetric about the diagonal. Each entry in the Table 5 is a fuzzy set, denoting the control action to be taken when the cause of action will be determined by the position of that particular entry in the table.

The normalized triangular fuzzy sets as given in Table 5 for the three different categories θ , $\dot{\theta}$ and the applied controlling force u are defined using the following function with adjustable parameters:

$$\mu(x) = \begin{cases} 1 - \frac{1}{c}|x - l|, & \text{if } l \leq x \leq l + 2c \\ 0 & \text{otherwise.} \end{cases} \quad (46)$$

The parameters of the membership function of the elements in the different category of fuzzy sets, as defined by (46) are given in Tables 3, 4, and 6. In Table 6 there are seven rows indicating seven linguistic labels. The last three

Table 3
Width of triangle for different categories of fuzzy sets

c-value	θ	$\dot{\theta}$	action
	0.025	0.0375	0.6

Table 4
Step length for the generation of three categories of fuzzy sets

h-value	θ	$\dot{\theta}$	action
	0.004	0.006	0.096

Table 5
The rule-base

	\wedge	NB	NM	NS	ZE	PS	PM	PB
Neg. Big:	NB	NB	NB	NB	NM	NS	NS	ZE
Neg. Medium:	NM	NB	NM	NM	NM	NS	ZE	PS
Neg. Small:	NS	NB	NM	NS	NS	ZE	PS	PS
Zero:	ZE	NM	NM	NS	ZE	PS	PM	PM
Pos. Small:	PS	NS	NS	ZE	PS	PS	PM	PB
Pos. Medium:	PM	NS	ZE	PS	PM	PM	PM	PB
Pos. Big:	PB	ZE	PS	PS	PM	PB	PB	PB

Table 6

Left-end points of the triangular distribution for different fuzzy sets under a single category

	l-value	θ	$\dot{\theta}$	Action
Neg. Big:	NB	−0.100	−0.1500	−2.4
Neg. Medium:	NM	−0.075	−0.1125	−1.8
Neg. Small:	NS	−0.050	−0.0750	−1.2
Zero:	ZE	−0.025	−0.0375	−0.6
Pos. Small:	PS	0.000	0.0000	0.0
Pos. Medium:	PM	0.025	0.0375	0.6
Pos. Big:	PB	0.050	0.0750	1.2

Table 7

Simulation results—centroid based defuzzification

step $\dot{\theta}$	sim-1 0.0	sim-2 0.075	sim-3 −0.1	sim-4 0.05	sim-5 −0.05
0.0	0.0900	0.0900	0.0900	0.0250	0.0250
0.4	0.0581	0.0781	0.0579	0.0170	0.0173
0.8	0.0355	0.0478	0.0354	0.0104	0.0105
1.0	0.0311	0.0364	0.0309	0.0086	0.0087
1.4	0.0232	0.0275	0.0231	0.0065	0.0065
1.8	0.0154	0.0197	0.0154	0.0057	0.0057
2.0	0.0118	0.0159	0.0118	0.0053	0.0053
2.4	0.0079	0.0096	0.0079	0.0044	0.0045
2.8	0.0062	0.0069	0.0062	0.0034	0.0035
3.0	0.0059	0.0063	0.0058	0.0030	0.0030
3.4	0.0052	0.0055	0.0051	0.0024	0.0024
3.8	0.0042	0.0048	0.0042	0.0022	0.0022
4.0	0.0037	0.0043	0.0037	0.0021	0.0021
4.4	0.0028	0.0033	0.0028	0.0019	0.0019
4.8	0.0024	0.0025	0.0024	0.0016	0.0016
5.0	0.0023	0.0024	0.0023	0.0015	0.0015
5.4	0.0021	0.0022	0.0021	0.0012	0.0012
5.8	0.0018	0.0020	0.0018	0.0009	0.0009
6.0	0.0017	0.0018	0.0017	0.0008	0.0008
6.4	0.0014	0.0016	0.0014	0.0007	0.0007
6.8	0.0011	0.0013	0.0011	0.0005	0.0005
7.0	0.0010	0.0012	0.0009	0.0004	0.0004
7.4	0.0008	0.0009	0.0007	0.0003	0.0003
7.8	0.0006	0.0008	0.0006	0.0002	0.0002
8.0	0.0005	0.0007	0.0005	0.0002	0.0002

columns in Table 6 represents three different linguistic variables, used in the process. There are, in all, twenty-one entries in this Table, representing the left-end point of the parametric triangular membership function. The three values in Table 3 represent the respective half-length of the base of each triangle formed under three different classes. In each case, the fuzzy sets are defined by membership values at fifty equi-spaced points, generated using the step-lengths listed in Table 4, from the respective universes of discourse, as stated.

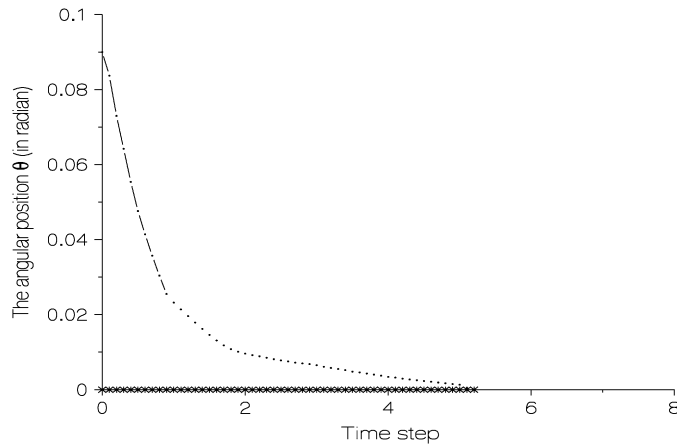
4.4. Results and discussion

The results of four simulations, following algorithm *Fuzzy Control* and using the specificity based defuzzification and centroid based defuzzification methods are presented in Tables 7 and 8. The variation of θ , the angular position of the pole, over time are also provided in the Figs. 3–7, for specificity based defuzzification method. The centroid based results are presented in Figs. 8–12. In all cases, initially the pole is given a small angular displacement in the range of $\pm 5^\circ$ together with a small angular velocity $\pm 10^\circ$ /per second and let it go. The results in Tables 7 and 8 show that the behaviour of the pendulum is consistently good.

Table 8

Simulation results—specificity based defuzzification

step	sim-1	sim-2	sim-3	sim-4	sim-5
$\dot{\theta}$	0.0	0.075	−0.075	0.0	−0.05
0.0	0.0900	0.0900	0.0900	0.0250	0.0250
0.2	0.0730	0.0745	0.0736	0.0221	0.0219
0.4	0.0554	0.0570	0.0562	0.0187	0.0184
0.6	0.0414	0.0446	0.0438	0.0153	0.0149
0.8	0.0304	0.0331	0.0327	0.0124	0.0122
1.0	0.0232	0.0242	0.0238	0.0103	0.0103
1.2	0.0196	0.0206	0.0202	0.0093	0.0093
1.4	0.0162	0.0171	0.0152	0.0086	0.0086
1.6	0.0131	0.0139	0.0122	0.0079	0.0079
1.8	0.0108	0.0112	0.0102	0.0073	0.0073
2.0	0.0096	0.0098	0.0093	0.0069	0.0069
2.2	0.0089	0.0090	0.0086	0.0063	0.0063
2.4	0.0081	0.0083	0.0078	0.0056	0.0056
2.6	0.0075	0.0076	0.0073	0.0051	0.0051
2.8	0.0070	0.0071	0.0068	0.0044	0.0044
3.0	0.0065	0.0066	0.0063	0.0038	0.0038
3.2	0.0058	0.0059	0.0056	0.0032	0.0033
3.4	0.0052	0.0053	0.0051	0.0028	0.0028
3.6	0.0046	0.0048	0.0044	0.0024	0.0024
3.8	0.0040	0.0041	0.0038	0.0020	0.0020
4.0	0.0034	0.0035	0.0033	0.0016	0.0016
4.2	0.0029	0.0030	0.0028	0.0007	0.0009
4.4	0.0025	0.0026	0.0024	0.0000	0.0000
4.6	0.0021	0.0022	0.0020	0.0000	0.0000
4.8	0.0017	0.0018	0.0016	0.0000	0.0000
5.0	0.0013	0.0014	0.0006	0.0000	0.0000
5.2	0.0000	0.0002	0.0000	0.0000	0.0000

Fig. 3. Specificity based controller characteristic: $\dot{\theta} = 0.0$.

5. Conclusion

The concept of similarity or indistinguishability is inherent in approximate reasoning methodology. We have studied at length different functions to measure the similarity between two inexact concepts. We have developed new measures for similarity between two fuzzy sets and appropriately introduced the concept in approximate reasoning methodology. It has been shown that this similarity based approximate reasoning technique is a combination of Zadeh's Compositional Rule of Inference and Turksen's similarity based reasoning. We have used this concept in the

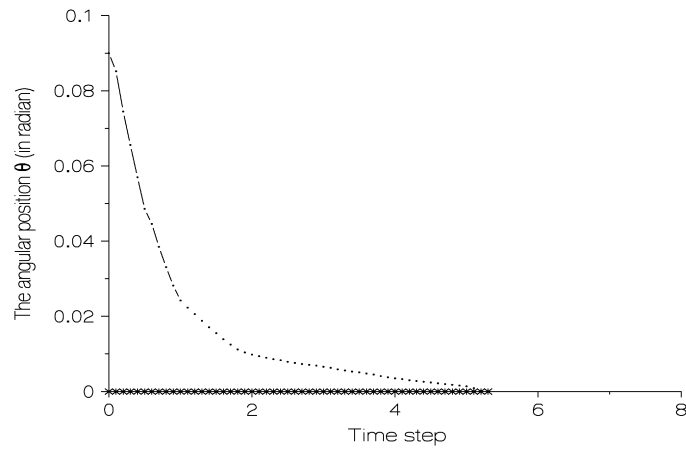


Fig. 4. Specificity based controller characteristic: $\dot{\theta} = 0.075$.

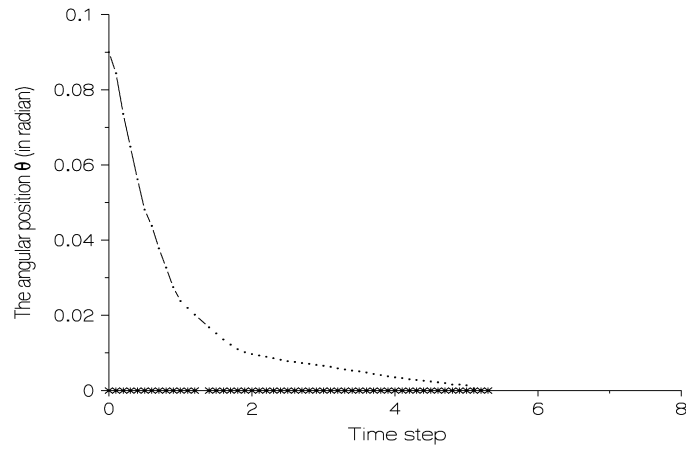


Fig. 5. Specificity based controller characteristic: $\dot{\theta} = -0.075$.

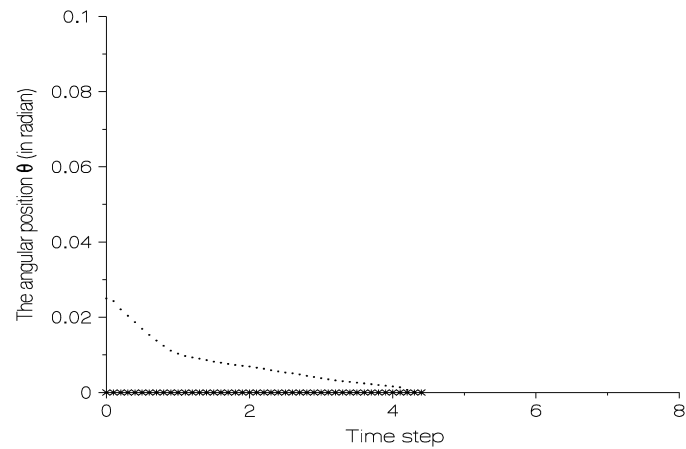


Fig. 6. Specificity based controller characteristic: $\dot{\theta} = 0.0$.

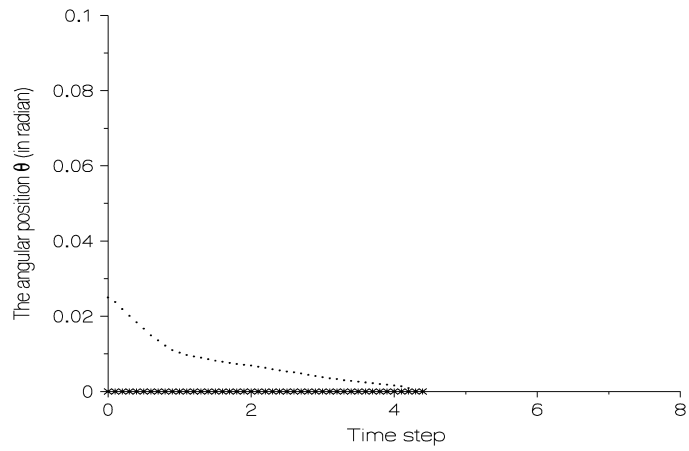


Fig. 7. Specificity based controller characteristic: $\dot{\theta} = -0.05$.

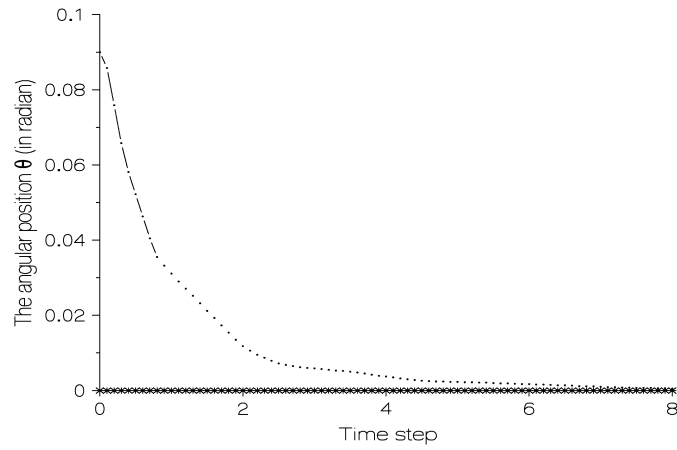


Fig. 8. Centroid based controller characteristic: $\dot{\theta} = 0.0$.

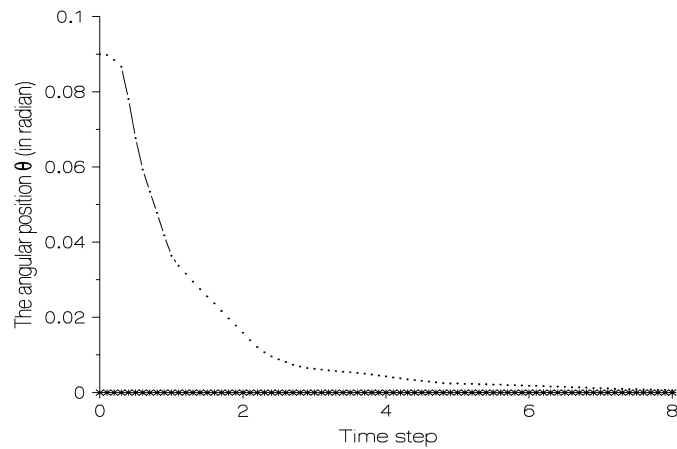
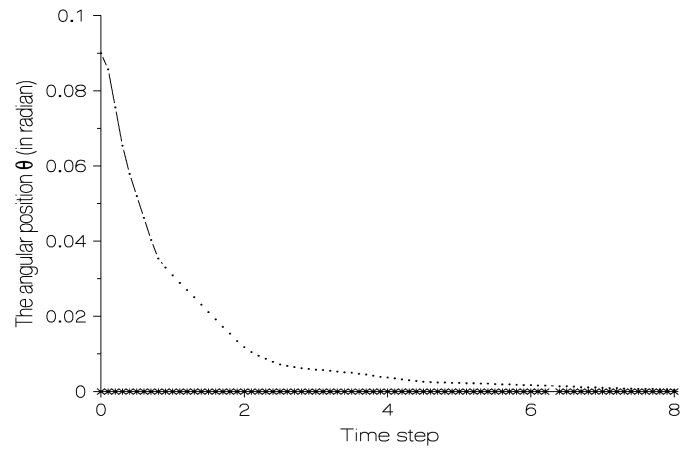
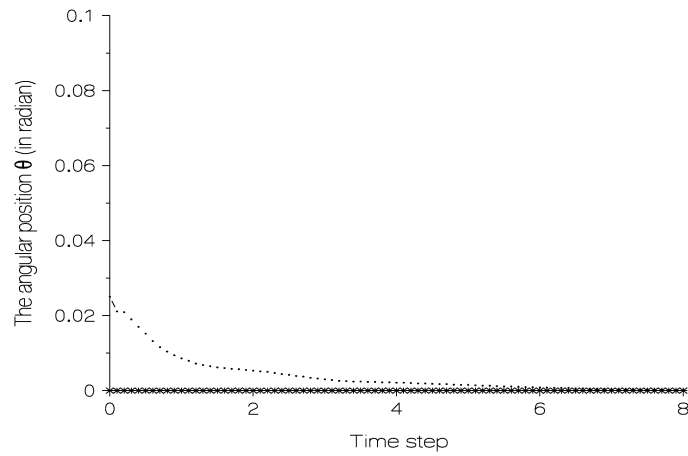
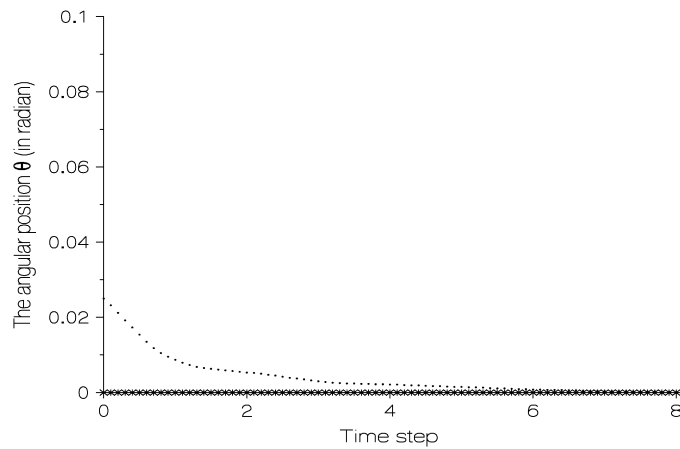


Fig. 9. Centroid based controller characteristic: $\dot{\theta} = 0.075$.

Fig. 10. Centroid based controller characteristic: $\dot{\theta} = -0.1$.Fig. 11. Centroid based controller characteristic: $\dot{\theta} = 0.05$.Fig. 12. Centroid based controller characteristic: $\dot{\theta} = -0.05$.

design of rule based fuzzy systems and then used it in rule based fuzzy control. Interesting results have been shown for a typical cart-pole problem.

We have suggested relevant issues involved in the design of fuzzy systems. Further research on the use of similarity based approximate reasoning is necessary to have a better understanding of the effect of the same on the cognitive process involved in fuzzy control. The concept of similarity is useful in fuzzy reasoning.

We have attempted to establish the fact that fuzzy set theory and similarity based approximate reasoning may be made more popular because of the scope of its application in wide and challenging fields of investigation.

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